TO DETERMINE THE GEODESIC CURVATURE OF A SURFACE CURVE BY ITS TRANSFORMED PLANE CURVE

by Hsieh Wan-chen

§ 1. Introduction

The generalization of the the criterion of parallelism with respect to a surface from that with respect to a plane has been given by Levi-Civita. From this point of view the geodesic curvature will be discussed as the curvature of a plane curve in this paper. This means that we can find the geodesic curvature from the curvature defined for the plane curve.

§ 2. Parallelism with respect to a surface

In Euclidean plane geometry, when two points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are fixed, then to every direction drawn from $\boldsymbol{P}_1$ there corresponds one and only one direction drawn from $\boldsymbol{P}_2$ and parallel to the first. That is the ordinary sense of parallelism with respect to a plane.

Let $\boldsymbol{P}_1$ be a point of surface S, $\boldsymbol{T}_1$ the corresponding tangent plane (at $\boldsymbol{P}_1$), and $\boldsymbol{R}_1$ an arbitrary tangent vector drawn from $\boldsymbol{P}_1$ and therefore lyinging in $\boldsymbol{T}_1$. Similary, $\boldsymbol{R}_2$ is a tangent vector drawn from another point $\boldsymbol{P}_2$ of S, and therefore lying in the corresponding tangent plane $\boldsymbol{T}_2$.

If S is developable, it can be defined that the two vectors $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are parallel, if they are parallel in the ordinary sense when S is developed upon a plane. This criterion fails in the case of a non-developable surface, such as a sphere, and it is natural to look for an adequate generalization of parallelism. Let $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ be connected by a specific curve $\boldsymbol{C}$ lying in S, then a developable surface S_T can be constructed by the tangent planes along $\boldsymbol{C}$. The tangent vectors $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ at $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are also tangential to S_T. Thus, it can be made the definition of surface parallelism on the non-developable S along C as the parallelism on S_T has just been defined. Of course, this parallelism depends on the specific curve $\boldsymbol{C}$.

As shown in Fig. 1, $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are parallel in the sense of Levi-Civita parallelism, $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are consecutive points of $\boldsymbol{C}$ on S, and $-\boldsymbol{w}$ is the infinitesimal vector representing the elementary rotation around the straight line $\boldsymbol{g}$ by means of which $\boldsymbol{T}_2$ is brought into coincidence with T_1. Thus, when R_1 takes the parallel displacement* from P_1 to P_2, it can be found

$$dR_1 = w \times R_1$$

As both w and R_1 are vectors in T_1, it follows that the increment is perpendicular to T_1, that is

$$dR_1 \parallel n \qquad (2\text{—}1)$$

where n is the normal to T_1. This is the condition for infinitesimal parallel displacement.

Knowing dR_1 is perpendicular to all directions of infinitesimal displacement in S, we can conclude that

$$dY^i \; \delta y^i = 0 \qquad (2\text{—}2)$$

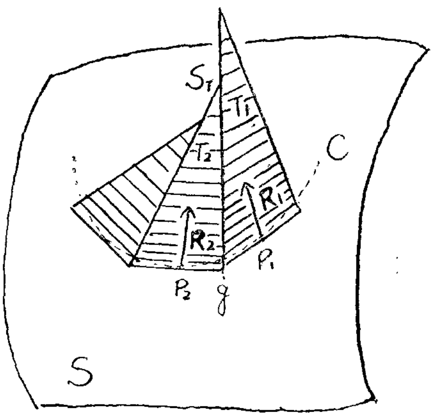

Fig. 1

where δy^i are the components of the infinitesimal vector of displacement and dY^i are the components of dR_1 in Cartesian coordinates.

Since δy^i are the components of a displacement along S, they can be expressed in term of the corresponding variations δu^α of the surface coordinates, that is

$$\delta y^i = \frac{\partial y^i}{\partial u^\alpha}\delta u^\alpha \quad \# \qquad (2\text{—}3)$$

As the vector R_1, with magnitude R, is tangential to S, it can be represented by

$$R^\alpha = R\lambda^\alpha \qquad (2\text{—}4)$$

where $\lambda^\alpha = \frac{du^\alpha}{ds}$ are parameters of the direction, then

$$Y^i = R\frac{dy^i}{ds} = R\frac{\partial y^i}{\partial u^\alpha}\frac{du^\alpha}{ds} = \frac{\partial y^i}{\partial u^\alpha}R^\alpha \qquad (2\text{—}5)$$

Putting

$$\tau_\alpha = \frac{\partial y^i}{\partial u^\alpha} d\left(\frac{\partial y^i}{\partial u^\beta}R^\beta\right) \qquad (2\text{—}6)$$

then the identity (2-2) can be written finally in the form

$$\tau_\alpha \; \delta u^\alpha = 0 \qquad (2\text{—}7)$$

Since δu^α are completely arbitrary, it follows from Eq. (2-7) that

$$\tau_\alpha = 0 \qquad (2\text{—}8)$$

Carrying out the differentiation of Eq. (2-6) and inserting $g_{\alpha\beta} = \frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^i}{\partial u^\beta}$, yields

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* Let the vector at P take a parallel motion along C, the transformed curve of C, to another point Q in the plane on which S_T was developed, then wrap the plane about S in its original form and position. The vector is said to have parallel displacement at its new position.

\# In general in what follows Latin indices take the values 1, 2, 3 and Greek indices the values 1, 2.

$$\tau_\alpha=\frac{\partial y^i}{\partial u^\alpha}\left[\frac{\partial y^i}{\partial u^\beta}dR^\beta+R^\gamma\frac{\partial^2 y^i}{\partial u^\gamma\partial u^\epsilon}du^\epsilon\right]$$
$$=g_{\alpha\beta}\ dR^\beta+[\gamma\epsilon,\ \alpha]\ R^\gamma\ du^\epsilon$$

where τ^α depend upon two vectors (R^α and the displacement du^α) as well as the coefficients of ds^2 and their first derivatives. So that it can be thought of as a covariant and its contravariant can be written in the form

$$\tau^\beta=\tau_\alpha g^{\alpha\beta}=dR^\beta+\left\{{\beta \atop \gamma\epsilon}\right\}R^\gamma\ du^\epsilon\ \ * \qquad (2\text{—}9)$$

It follows from Eq. (2-8) and Eq. (2-9) that the condition (2—1) is equivalent to

$$\frac{\delta R^\beta}{\delta s}=\frac{dR^\beta}{ds}+R^\gamma\left\{{\beta \atop \gamma\epsilon}\right\}\frac{du^\epsilon}{ds}=o \qquad (2\text{—}10)$$

where $\frac{\delta R^\beta}{\delta s}$ is the intrinsic derivative of R^β

§3. Geodesic curvature of a surface curve and the curvature of its transformed plane curve

Let equations of the curve C lying in the surface

$$S:\qquad x^i=x^i,\ (u^1,\ u^2) \qquad (3\text{—}1)$$

be given in the form

$$C:\qquad u^\alpha=u^\alpha\ (s) \qquad (3\text{—}2)$$

where s is the arc parameter.

Let a system of unit vector R^α (s) be parallel with respect to C. The angle θ, in Levi-Civita sense, made by R^α and the unit tangent vector λ^α (s) at each point of C can be written as

$$\sin\ \theta=\varepsilon_{\alpha\beta}\ \lambda^\alpha\ R^\beta \qquad (3\text{—}3)$$

Differentiating intrinsically the Eq. (3-3), and then substituting Eq. (2—10), $\cos\theta=g_{\alpha\beta}\ \lambda^\alpha\ R^\beta$ and $\frac{\delta\epsilon^{\alpha\beta}}{\delta s}=o$, gives

$$\lambda_\beta R^\beta\ \frac{d\theta}{ds}=\varepsilon_{\alpha\beta}\ R^\beta\frac{\delta\lambda^\alpha}{\delta s}$$

or

$$R^\beta\ \left(\lambda_\beta\frac{d\theta}{ds}-\varepsilon_{\alpha\beta}\frac{\delta\lambda^\alpha}{\delta s}\right)=o \qquad (3\text{—}4)$$

Since the angles formed by two families of parallel vectors along a curve C are equal at each point, the expression in the parentheses of Eq. (3—4) is independent of the choice of R^α, and consequently

$$\frac{d\theta}{ds}=\varepsilon_{\alpha\beta}\frac{\delta\lambda^\alpha}{\delta s}\lambda^\beta \qquad (3\text{—}5)$$

If only the numerical value of $\frac{d\theta}{ds}$ is considered, Eq. (3-5) can be written as

* $[\gamma\epsilon,\ \alpha]$ are the Christoffel symbol of the first kind, and $\left\{{\beta \atop \gamma\epsilon}\right\}$ are the second kind.

$$\left|\frac{d\theta}{ds}\right| = e\ \varepsilon_{\alpha\beta}\frac{\delta\lambda^\beta}{\delta s}\lambda^\alpha$$

$$= e\ \varepsilon_{\alpha\beta}\left(\frac{d^2u^\beta}{ds^2}+\left\{\begin{matrix}\beta\\ \gamma\epsilon\end{matrix}\right\}\frac{du^\gamma}{ds}\frac{du^\epsilon}{ds}\right)\frac{du^\alpha}{ds} \qquad (3\text{—}6)$$

where e always makes $\varepsilon^{\alpha\beta}\frac{\delta\lambda^\beta}{\delta s}\lambda^\alpha$ positive.

Now let λ^i be the space components of the tangent vector λ^α, then

$$\lambda^i=\frac{dx^i}{ds}=\frac{\partial u^i}{\partial u^\alpha}\frac{du^\alpha}{ds}=x^i_\alpha\ \lambda^\alpha \qquad (3\text{—}7)$$

Differentiating Eq. (3-7) intrinsically with respect to s, gives

$$\frac{\delta\lambda^i}{\delta s}=x^i_{\alpha,\beta}\ \lambda^\alpha\frac{du^\beta}{ds}+x^i_\alpha\frac{\delta\lambda^\alpha}{\delta s} \qquad (3\text{—}8)$$

Put

$$\frac{\delta\lambda^\alpha}{\delta s}=\rho\eta^\alpha\ ,\quad \frac{\delta\lambda^i}{\delta s}=\mathrm{K}\ \mu^i \qquad (3\text{—}9)$$

where ρ is a suitable scalar that mekes η^α a unit vector, K is the curvature and both η^α and μ^i are perpendicular to λ^α.

Substitution of Eq. (3-9) and $x^i_{\alpha,\beta}=b_{\alpha\beta}\ n^i$ (Gauss's formula) into Eq. (3-8), gives

$$\mathrm{K}\mu^i=b_{\alpha\beta}\ \lambda^\alpha\ \lambda^\beta\ n^i+\rho\eta^i \qquad (3\text{—}10)$$

where n^i is the unit vector normal to S at P and $\eta^i=x^i_\alpha\ \eta^\alpha$. Multipling both sides of Eq. (3-10) by η_i and substituting $n^i\eta_i=o$, yields

$$\mathrm{K}\ \mu^i\ \eta_i=\rho\eta^i\eta_i$$

or

$$\mathrm{K}\ sin\ \varphi=\rho \qquad (3\text{—}11)$$

in which φ is the angle between μ^i and n^i.

From Eq. (3-9) and Eq. (3-11), Eq. (3-6) can be found as

$$\left|\frac{d\theta}{ds}\right|=e\ \varepsilon_{\alpha\beta}\ \eta^\alpha\ \lambda^\beta\ \mathrm{K}\ sin\ \varphi$$

$$=e\ \mathrm{K}\ sin\ \varphi=e\ \mathrm{K}_g \qquad (3\text{—}12)$$

Note that K_g is the geodesic curvature of C at P by definition, therefore Eq. (3-12) means that the rate of change of the unit tangent λ^α with respect to s subject to the Levi-Civita sense of parallelism, and the geodesic curvature are equal in numerical value. From §2 and the definition of curvature of the plane curve, $\frac{d\theta}{ds}$ can be considered as the curvature of the plane curve $\bar{C}$ which is the transformed curve of C by the Levi-Civita method. Hence from this viewpoint, the geodesic curvature of C can be found by means of the curvature of $\bar{C}$, if its sign is neglected.

§4. Illustrative examples

1. The curve lies on a developable surface.

A circular cylinder

$$y^1=a\ (1+\cos 2u)\ ,\quad y^2=a\sin 2u\ ,\quad y^3=v \qquad (4\text{—}1)$$

intersects the sphere

$$y^1=2a\sin\alpha\cos\beta,\ y^2=2a\sin\alpha\sin\beta,\ y^3=2a\cos\beta$$

at the curve C whose equations in the first octant are

$$y^1=a\ (1+\cos 2\varphi),\ y^2=a\sin 2\varphi,\ y^3=2a\sin\varphi \qquad (4\text{—}2)$$

where $\quad o\leqq\varphi\leqq\frac{\pi}{2}.$

If C is considered as a curve on the cylinder (4-1), find the geodesic curvature of C.

Solution: Developing the cylinder (4-1) upon the xy-plane in such a way that the semi-circle C' (see Fig. 2) maps into the positive x-axis with the origin corresponding to P, yields the transformation

$$x=2a\varphi\ ,\quad y=2a\sin\varphi \quad (o\leqq\varphi\leqq\frac{\pi}{2}) \qquad (4\text{—}3)$$

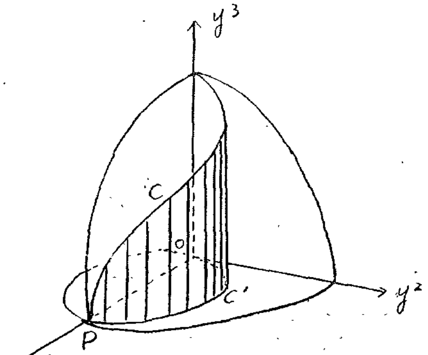

Fig. 2

which is the equation of the plane curve $\bar{C}$ corresponding to the surface curve C. Then the geodesic curvature of C found from Eq. (3-12) is the curvature of $\bar{C}$, i.e.

$$eK_g=\left|\frac{d\theta}{ds}\right|=\left|\frac{\frac{dx}{d\varphi}\frac{d^2y}{d\varphi^2}-\frac{dy}{d\varphi}\frac{d^2x}{d\varphi^2}}{[(\frac{dx}{d\varphi})^2+(\frac{dy}{d\varphi})^2]^{\frac{3}{2}}}\right|$$

$$=\left|\frac{-4a^2\sin\varphi}{(4a^2+4a^2\cos^2\varphi)^{\frac{3}{2}}}\right|=\frac{\sin\varphi}{2a\ (1+\cos^2\varphi)^{\frac{3}{2}}} \qquad (4\text{—}4)$$

It is easy to check this result from the original definition. Differentiating Eq. (4-2) with respect to s, yields

$$\frac{dy^1}{ds}=-2a\sin 2\varphi\frac{d\varphi}{ds}\ ,\ \frac{dy^2}{ds}=2a\cos 2\varphi\frac{d\varphi}{ds}\ ,\ \frac{dy^3}{ds}=2a\cos\varphi\frac{d\varphi}{ds} \qquad (4\text{—}5)$$

From Eq. (4—5), ds can be found as

$$ds=2a\ (1+\cos^2\varphi)^{\frac{1}{2}}d\varphi$$

And hence

$$\frac{du}{ds}=\frac{1}{2a\ (1+\cos^2\varphi)^{\frac{1}{2}}}\ ,\quad \frac{dv}{ds}=\frac{\cos\varphi}{(1+\cos^2\varphi)^{\frac{1}{2}}},$$

$$\frac{d^2u}{ds^2}=\frac{\sin\varphi\cos\varphi}{4a^2\ (1+\cos^2\varphi)^2}\ ,\quad \frac{d^2v}{ds^2}=\frac{-\sin\varphi}{2a\ (1+\cos^2\varphi)^2} \qquad (4\text{—}6)$$

The coefficients of the first fundamental form of the cylinder (4-1) can be found to be

$$g_{11}=4a^2\ ,\quad g_{12}=g_{21}=o\ ,\quad g_{22}=1 \qquad (4\text{—}7)$$

Substituting Eq. (4-6) and Eq. (4-7) into Eq. (3-6) and noting that $\left\{\begin{matrix}\alpha\\ \gamma\epsilon\end{matrix}\right\}=o$, $\lambda^\alpha=\frac{du^\alpha}{ds}$, gives

$$eK_g=e\ (\varepsilon_{12}\frac{d^2v}{ds^2}\frac{du}{ds}+\varepsilon_{21}\frac{d^2u}{ds^2}\frac{dv}{ds})$$

$$=e\left(\sqrt{g}\frac{-\sin\varphi}{4a^2(1+\cos^2\varphi)^{\frac{5}{2}}}-\sqrt{g}\frac{\sin\varphi\cos^2\varphi}{4a^2(1+\cos^2\varphi)^{\frac{5}{2}}}\right)$$

$$=e\frac{-\sin\varphi}{2a(1+\cos^2\varphi)^{\frac{3}{2}}}=\frac{\sin\varphi}{2a(1+\cos^2\varphi)^{\frac{3}{2}}} \qquad (4\text{—}8)$$

Noting that Eq. (4-8) and Eq. (4-4) are the same, this is the result.

2. The curve on the non-developable surface.

Find the geodesic curvature of the small circle

$$C:\ y^1=a\sin\varphi\cos\psi\ ,\quad y^2=a\sin\varphi\sin\psi\ ,\quad y^3=a\cos\varphi \qquad (4\text{—}9)$$

(φ=constant) on the sphere

$$S:\ y^1=a\sin\varphi\cos\psi\ ,\quad y^2=a\sin\varphi\sin\psi\ ,\quad y^3=a\cos\varphi \qquad (4\text{—}10)$$

Solution: As shown in Fig. 3, the developable surface S_T made by the tangent planes along C is a right circular cone. If S_1 is developed upon a plane, the small circle C maps into the curve

$$\bar{C}:\ x=a\tan\varphi\cos w\ ,\quad y=a\tan\varphi\sin w \qquad (4\text{—}11)$$

where $w=\psi\cos\varphi$, $\varphi=constant$, $o\leqq w\leqq 2\pi\cos\varphi$.

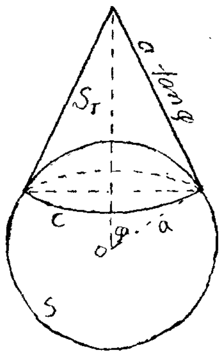

Fig. 3

The curvature of $\bar{C}$ is the geodesic curvature of C as discussed in §3, such that, from Eq. (4-11)

$$eK_g=\left|\frac{d\theta}{ds}\right|=\left|\frac{\frac{dx}{dw}\frac{d^2y}{dw^2}-\frac{dy}{dw}\frac{d^2x}{dw^2}}{\left[\left(\frac{dx}{dw}\right)^2+\left(\frac{dy}{dw}\right)^2\right]^{\frac{3}{2}}}\right|=\frac{a^2\tan^2\varphi}{(a\tan\varphi)^3}=\frac{1}{a\tan\varphi} \qquad (4\text{—}12)$$

Now let us find K_g of C directly from the right side of Eq. (3-6). From Eq. (4-10), it is easy to find

$$g_{11}=a^2\sin^2\varphi\ ,\quad g_{12}=o\ ,\quad g_{22}=a^2 \qquad (4\text{—}13)$$

Using Eq. (4-13) and noting that φ=constant on the curve C gives, from Eq. (3-6),

$$eK_g=e\varepsilon_{\alpha\beta}\left(\frac{d^2u^\beta}{ds^2}+\begin{Bmatrix}\beta\\ \gamma\epsilon\end{Bmatrix}\frac{du^\gamma}{ds}\frac{du^\epsilon}{ds}\right)\frac{du^\alpha}{ds}=e\left[\varepsilon_{12}\left(\frac{d^2\varphi}{ds^2}+\begin{Bmatrix}2\\ \gamma\epsilon\end{Bmatrix}\frac{du^\gamma}{ds}\frac{du^\epsilon}{ds}\right)\frac{d\psi}{ds}\right.$$

$$\left.+\varepsilon_{21}\left(\frac{d^2\psi}{ds^2}+\begin{Bmatrix}1\\ \gamma\epsilon\end{Bmatrix}\frac{du^\gamma}{ds}\frac{d\varphi^\epsilon}{ds}\right)\frac{d\varphi}{ds}\right]=e\sqrt{g}\begin{Bmatrix}2\\ \gamma\epsilon\end{Bmatrix}\frac{du^\gamma}{ds}\frac{du^\epsilon}{ds}\frac{d\psi}{ds}$$

$$=e\sqrt{g}\begin{Bmatrix}2\\ 11\end{Bmatrix}\left(\frac{d\psi}{ds}\right)^3=e\sqrt{g}\,g^{2\alpha}[11,\alpha]\left(\frac{d\psi}{ds}\right)^3$$

$$=e\frac{g_{11}}{\sqrt{g}}[11,2]\left(\frac{d\psi}{ds}\right)^3=e\left(\frac{a^2\sin^2\varphi}{a^2\sin\varphi}\right)(-a^2\sin\varphi\cos\varphi)\left(\frac{1}{a\tan\varphi}\right)^3$$

$$=\frac{1}{a\tan\varphi} \qquad (4\text{—}14)$$

This is the same result as Eq. (4-12).

It is clear that this result can be found more easily by geometrical method from Fig. 3.

§ 5. Conclusion and discussion

In special cases when Eq. (3-2) is a plane curve, then $\bar{C}$ is identical with C. If $u^1=x$, $u^2=y$ are taken, then $g_{11}=g_{22}=1$, $g_{12}=o$; $\left\{ {\beta \atop \gamma\epsilon} \right\}=o$; and the Eq. (3-6) becomes

$$\left| \frac{d\theta}{ds} \right| = e \left(\frac{dx}{ds} \frac{d^2y}{ds^2} - \frac{dy}{ds} \frac{d^2x}{ds^2} \right) \qquad (5\text{—}1)$$

It is evident that Eq. (5-1) is the formula of the curvature of the plane curve.

We have seen from § 3 that the determination of the geodesic curvature of a surface curve reduces to that of the curvature of its transformed plane curve, so that the determination of the geodesic curvature of a surface curve needs only the developed plane curve $\bar{C}$.

It is evident that the illustrative examples are selected for the sake of easiness and the calculations are simpler than usual. In general, the suitable transformation between C and $\bar{C}$ should be found in advance.

References:

1. I. S. Sokolnikoff: Tensor Analysis.
2. L.P. Eisenhart: An Introduction to Differential Geometry.
3. D. J. Struik: Lectures on Classical Differential Geometry.
4. Tullo Levi-Civita: The Absolute Differential Calculus.

藉轉換之平面曲線測定曲面曲線之測地曲率

解 萬 臣

Levi-Civita 曾將平面上之平行觀念推廣至曲面上而得一曲面上之平行定義，有關曲面曲線 (surface curve) 之研究自可盡量利用之。本文得用該氏觀念導出求任一曲面曲線之測地曲率之間接法則，其步驟係作此曲面曲線之有關可展曲面次將此可展曲面展爲平面而得一對應之平面曲線，最後由此平面曲線之曲率而求得該曲面曲線之測地曲率。